Mathematical Statements and basic logic (Also see "mathematical hygiene")

A statement is a sentence that is true or false but not both (i.e. it has a truth value)

Examples

- Today is August 31, 2017
- $\frac{1}{2} = \frac{3}{6}$
- $\pi \in \mathbb{R}$
- It rained yesterday
- The sum of two odd integers is odd.

Sometimes, we use letters in place of statements:

P = the product of two integers is always odd

then we say "P is False"

Negation

The negation of a statement p is $\neg p$ ("not p"). It is false when P is true and true when P is false.

Examples

- P = (7 is odd), $\neg P$ = (7 is not odd) or (7 is even)
- P = All integers are prime, $\neg P$ = (not all integers are prime) or (some integers are not prime)
- Some cows are brown, $\neg P$ = no cows are brown

A truth table shows the possible truth values of a logical operation.

Here's the truth table for negation:

P	$\neg P$
T	F
F	T

Equivalent statements

For any $x \in \mathbb{R}$, the statements "$x^2 > 0$" and "$x \neq 0$" are equivalent.

"Not all cows are brown" is equivalent to "some cows are not brown."

Two statements are equivalent if they have the same truth values.
If P is equivalent to Q, we write $P \Leftrightarrow Q$.

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Note: $P \Leftrightarrow \neg(\neg P)$

And/or

P and Q $(P \wedge Q)$ is true when both P and Q are true.

(P = 7 is odd, Q = 7 is prime, $P \wedge Q$ = 7 is odd + prime)

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P or Q $(P \vee Q)$ is true when P is true or Q is true (or both)

(P= 7 is odd, Q = 7 is prime, $P \vee Q$ = "7 is odd or prime" is true)

Caution: This is different than the standard English meaning.

Truth table for or:

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

(<u>Note</u>: $P \vee \neg P$ is always true!)

<u>De Morgan's Laws</u>

Negation of $P \wedge Q$?

e.g. "I bought carrots <u>and</u> lettuce"

P = I bought carrots

Q = I bought lettuce

Negation is "Either I didn't buy carrots or I didn't buy lettuce"

(Why not "I bought neither carrots nor lettuce"?)

$\boxed{\text{De Morgan \# 1:}}$ $\neg(P \wedge Q) \Leftrightarrow (\neg P) \vee (\neg Q)$

<u>Pf</u>: If $\neg(P \wedge Q)$ is true, then $P \wedge Q$ is false. Thus, either P is false or Q is false, Which means $\neg P$ is true or $\neg Q$ is true. In other words, $(\neg P) \vee (\neg Q)$ is true.

On the other hand, if $\neg(P \wedge Q)$ is false, then $P \wedge Q$ is true, which means that P is true and Q is true. In other words, $\neg P$ is not true and $\neg Q$ is not true. Thus $(\neg P) \vee (\neg Q)$ must be false. $\square$

Negation of $P \vee Q$?

e.g. "You can't rent this apartment if you have a cat or a dog."

P = You can rent this apartment

Q = You have a cat

R = You have a dog

Statement is: $\underbrace{(Q \vee R)}_{\text{You have a cat or a dog}} \Rightarrow \underbrace{\neg P}_{\text{You can't rent this apartment}}$

$\neg(Q \vee R)$ = "You don't have a cat <u>and</u> you don't have a dog."

$= (\neg Q) \wedge (\neg R)$

De Morgan #2: $\neg(Q \vee R) \Leftrightarrow (\neg Q) \wedge (\neg R)$

<u>Pf</u>: In homework #1.

If/then statements

<u>Ex:</u> If it's raining tomorrow, I won't ride my bike to school.

P = It's raining tomorrow

Q = I won't ride my bike to school (tomorrow)

(or R = I will ride my bike to school)

Statement can be written as: $P \Rightarrow Q$ ("If P, then Q" or "P implies Q")

(or $P \Rightarrow \neg R$)

Truth table:

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

<u>Note</u>: If P is <u>false</u> then $P \Rightarrow Q$ will be true for <u>any</u> Q!

<u>Examples:</u>

- If the sun doesn't set tonight, then I'll run a marathon tomorrow.
- "I'll do _____ when pigs fly"
- If $0 = 1$, then $7 > 9$.

The only time $P \Rightarrow Q$ is <u>false</u> is if P is true and Q is false.

e.g. if it rains tomorrow <u>and</u> I ride my bike to school

i.e. $\neg(P \Rightarrow Q) \Longleftrightarrow P \wedge (\neg Q)$

Check w/ truth table

P	Q	$\neg(P \Rightarrow Q)$	$P \wedge (\neg Q)$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	F	F

e.g. the negation of "If it's above 80 degrees, I'm wearing shorts" is "It's above 80 degrees and I'm not wearing shorts."

<u>Note</u>: $(P \Leftrightarrow Q)$ is equivalent to $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$

Can write as "P if and only if Q"

Predicates

The truth of some statements depends on an input

e.g. $x^2 < 2$ is either true or false, depending on the value of x. Statement is called a predicate.

Can write $P(x) = (x^2 < 2)$.

Then $P(0)$ is true, $P(2)$ is false.

$\neg P(x) = (x^2 \geq 2)$

Quantifiers

Can build statements from predicates using quantifiers "there exists" and "for all".

Examples:

- "There exists $x \in \mathbb{Z}$ such that $x^2 < 2$"
  (or "$\exists\ x \in \mathbb{Z}$ s.t. $x^2 < 2$")

  Negation: "For all $x \in \mathbb{Z}$, $x^2 \geq 2$."

  ("$\forall\ x \in \mathbb{Z}$, $x^2 \geq 2$") ← obviously false, but a valid statement.

- "Every rational number is a real number" can be written
  "$\forall\ x \in \mathbb{Q}$, $x \in \mathbb{R}$".

Negation? "$\exists\ x \in \mathbb{Q}$ s.t. $x \notin \mathbb{R}$" (again, false!)

- "Every real number has a $\overset{\text{real}}{\text{number}}$ smaller than it."

  can be written "$\forall\ x \in \mathbb{R},\ \exists\ y \in \mathbb{R}$ s.t. $y < x$."

  Negation "$\exists\ x \in \mathbb{R}$ s.t. $\forall\ y \in \mathbb{R}\ \ y \geq x$."

- "All dogs are brown" Negation: "There exists a dog that is not brown."

  Note: The negation is not "No dogs are brown."
  Do you see why?

In general:

$$\neg\left(\forall\ x,\ P(x)\right) \iff \exists\ z \text{ s.t. } \neg P(z)$$

and $\neg\left(\exists\ x \text{ s.t. } P(x)\right) \iff \forall\ z,\ \neg P(z)$.